

# FIVE-TERM RELATION AND MACDONALD POLYNOMIALS

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ABSTRACT. The non-commutative five-term relation  $T_{1,0}T_{0,1} = T_{0,1}T_{1,1}T_{1,0}$  is shown to hold for certain operators acting on symmetric functions. The “generalized recursion” conjecture of Bergeron and Haiman is a corollary of this result.

## 1. INTRODUCTION

Suppose we have a family of operators  $\{R_{m,n}\}$  acting on some vector space  $V$ , where  $m, n$  run over all pairs of integers  $m, n \geq 0$ . Suppose  $R_{0,0}$  is the identity operator. For each pair of relatively prime integers  $m, n \geq 0$  let

$$T_{m,n} = \sum_{k=0}^{\infty} u^{mk} v^{nk} R_{mk,nk} \in \text{End}[V][[u, v]].$$

We say that the *five-term relations* hold if for any integers  $m, n, m', n' \geq 0$  such that  $mn' - m'n = 1$  we have:

$$(1) \quad T_{m,n} T_{m',n'} = T_{m',n'} T_{m+m',n+n'} T_{m,n}.$$

For instance, for  $(m, n) = (1, 0)$  and  $(m', n') = (0, 1)$  we obtain

$$(2) \quad T_{1,0} T_{0,1} = T_{0,1} T_{1,1} T_{1,0}.$$

The motivation for the name comes from the fact that Faddeev-Kashaev’s quantum dilogarithm satisfies a very similar identity (see [FK94], where it is shown to be a deformation of the classical five-term relation for the Roger’s dilogarithm.) In [KS08] one can find the identity (2)<sup>1</sup>, where it is related to wall-crossing for Donaldson-Thomas invariants.

Using relations (1) the operator  $T_{m,n}$  for any  $m, n$  can be expressed as a composition using  $T_{0,1}$ ,  $T_{1,0}$  and their inverses. Thus we obtain, in particular, that  $R_{m,n}$  for any  $m, n \in \mathbb{Z}_{\geq 0}$  belongs to the algebra generated by the elements of the form  $R_{0,k}$  and  $R_{k,0}$  ( $k \in \mathbb{Z}_{>0}$ ). Then (1), when expressed in terms of these elements, provide some interesting relations between them. For instance, taking the coefficient of  $u^k v$

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<sup>1</sup>The identities (1) are also valid there due to the obvious  $SL_2(\mathbb{Z})$  invariance

in both sides of (2) we obtain

$$[R_{k,0}, R_{0,1}] = R_{1,1} R_{k-1,0}.$$

For  $k = 1$  we have  $[R_{1,0}, R_{0,1}] = R_{1,1}$ , thus in general

$$[R_{k,0}, R_{0,1}] = [R_{1,0}, R_{0,1}] R_{k-1,0}.$$

It is an interesting problem to describe a complete set of relations between the operators  $R_{0,k}$ ,  $R_{k,0}$  which is in some sense smaller than the original set implied by (1).

Here we prove that our relations are satisfied by certain operators acting on the space of symmetric functions in infinitely many variables over the field  $\mathbb{Q}(q, t)$ . In fact, we have two statements. In the first statement we have

$$(3) \quad R_{k,0} = h_k^\perp, \quad R_{0,k} = (-1)^k \Delta'_{e_k},$$

where the operator  $h_k^\perp$  is the operator conjugate to the operator of multiplication by  $h_k$  with respect to the Hall inner product,  $\Delta'_F$  for a symmetric function  $F$  denotes the operator defined in the basis of the modified Macdonald polynomials as follows:

$$\Delta_F \tilde{H}_\lambda = F[B_\lambda], \quad B_\lambda = \sum_{c,r \in \lambda} q^c t^r,$$

$$\Delta'_F = \Delta_{F'} \quad \text{for } F'[X] = F[-1/M + X].$$

where  $M = (1 - q)(1 - t)$ . In the second statement

$$(4) \quad R_{k,0} = (-1)^k \Delta'_{e_k}, \quad R_{0,k} = (-1)^k \underline{e}_k \left[ \frac{X}{M} \right],$$

where  $\underline{e}_k \left[ \frac{X}{M} \right]$  denotes the operator of multiplication by  $e_k \left[ \frac{X}{M} \right]$ . The two statements are related by the conjugation with respect to the modified Hall inner product.

**Theorem 1.1.** *The operators defined by (3) (alternatively, by (4)) extend to a family of operators  $\{R_{m,n}\}$  satisfying the five-term relations (1).*

Bergeron and Haiman conjectured ([BH13], Conjecture 6) certain identities between the  $\Delta_{e_k}$  and  $h_k^\perp$  operators, which imply interesting recursion relations for the Macdonald polynomials. We show that their statement follows by expanding (2). In fact, it is our attempt to prove Conjecture 6 that led us to the discoveries of the present work.

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## 2. THE PROOF

We introduce extra variables  $u$  and  $v$  and set

$$\tau_u = \sum_{n=0}^{\infty} u^n h_n^\perp, \quad \Delta_v = \sum_{n=0}^{\infty} (-v)^n \Delta_{e_n}, \quad \Delta'_v = \sum_{n=0}^{\infty} (-v)^n \Delta'_{e_n} = \text{Exp}[v/M] \Delta_v,$$

then we have, for any symmetric function  $F$ ,

$$(\tau_u F)[X] = F[X + u], \quad \Delta_v \tilde{H}_\lambda = \prod_{r,c \in \lambda} (1 - vq^c t^r) \tilde{H}_\lambda,$$

$$\Delta_v^{-1} = \sum_{n=0}^{\infty} v^n \Delta_{h_n}, \quad \Delta_v^{-1} \tilde{H}_\lambda = \prod_{r,c \in \lambda} \frac{1}{1 - vq^c t^r} \tilde{H}_\lambda.$$

It is convenient to modify the operator  $\nabla$  by adding a sign, so that it becomes closer to  $\Delta_v$  in shape:

$$\nabla H_\lambda = \prod_{r,c \in \lambda} (-q^c t^r) H_\lambda = (-1)^{|\lambda|} q^{n'(\lambda)} t^{n(\lambda)} H_\lambda.$$

Now we write the identity (77) of [BH13], which is equivalent to Conjecture 6 there, in a generating function form. It claims

$$\nabla^{-1} h_k^\perp \nabla h_l^\perp = \sum_{r=0}^k (-1)^{k-r} \Delta_{h_r} h_{k+l}^\perp \Delta_{e_{k-r}} \quad (k, l \in \mathbb{Z}_{\geq 0}).$$

Remember that our  $\nabla$  has a sign that conveniently turns  $(-1)^r$  to  $(-1)^{k-r}$ . Now we multiply both sides by  $u^k v^l$  and sum up:

$$\nabla^{-1} \tau_u \nabla \tau_v = \sum_{\substack{r \leq k \\ k, l, r \geq 0}} (-1)^{k-r} \Delta_{h_r} h_{k+l}^\perp \Delta_{e_{k-r}} u^k v^l.$$

We change the indexing:  $i = k + l$ ,  $j = k - r$ , so that the new summation runs over  $i, j, r \geq 0$ ,  $i \geq j + r$ . We obtain (the old indexes are  $k = j + r$ ,  $l = i - (j + r)$ ):

$$\sum_{j, r \geq 0, i \geq j+r} (-1)^j \Delta_{h_r} h_i^\perp \Delta_{e_j} u^{j+r} v^{i-(j+r)} = \Delta_{u/v}^{-1} \tau_v \Delta_{u/v} \Big|_{v \geq 0},$$

where the notation  $\Big|_{v \geq 0}$  stands for “keep only the terms with non-negative power of  $v$ ”.

**Proposition 2.1** (Generating function form of the conjecture). *Conjecture 6 of [BH13] is equivalent to*

$$(5) \quad \nabla^{-1} \tau_u \nabla \tau_v = \Delta_{u/v}^{-1} \tau_v \Delta_{u/v} \Big|_{v \geq 0}$$

Now we have the following observation easily deduced from the Pieri rules for Macdonald polynomials:

**Proposition 2.2.** *For a symmetric function  $F$  of degree  $d$  the operator*

$$\Delta_v^{-1} F^\perp \Delta_v$$

*is a polynomial in  $v$  of degree  $\leq d$ . The coefficient of  $v^d$  is  $\nabla^{-1} F^\perp \nabla$ .*

This implies that in  $\Delta_{u/v}^{-1} \tau_v \Delta_{u/v}$  the exponent of  $u/v$  is less or equal to the exponent of  $v$ . Therefore we don't have any terms with negative powers of  $v$  and we can omit  $\Big|_{v \geq 0}$  from (5). So the conjecture is equivalent to the following:

$$\nabla^{-1} \tau_u \nabla \tau_v = \Delta_{u/v}^{-1} \tau_v \Delta_{u/v}.$$

Now we move  $\tau_v$  to the right, substitute  $uv$  in  $u$  and interchange  $u$  and  $v$ :

$$\nabla^{-1} \tau_{uv} \nabla = \Delta_v^{-1} \tau_u \Delta_v \tau_u^{-1}.$$

This is what we are going to prove. We write the right hand side as follows:

$$(6) \quad \sum_{i,j \geq 0} W_{i,j} u^i v^j := \Delta_v^{-1} \tau_u \Delta_v \tau_u^{-1}$$

The main idea is to put parentheses in the right hand side in two different ways.

**2.1. First way.**  $(\Delta_v^{-1} \tau_u \Delta_v) \tau_u^{-1}$ . By Proposition 2.2 we know that in each non-zero term the exponent of  $v$  is less or equal to the exponent of  $u$ . Multiplying by  $\tau_u^{-1}$  only increases the exponents of  $u$ . Moreover, to calculate the terms where the exponent of  $v$  equals the exponent of  $u$  we can replace  $\Delta$  by  $\nabla$ . Therefore we have

**Proposition 2.3.** *We have  $W_{i,j} = 0$  for  $i < j$  and  $W_{i,i} = \nabla^{-1} h_i^\perp \nabla$ .*

**2.2. Second way.**  $\Delta_v^{-1} (\tau_u \Delta_v \tau_u^{-1})$ . It turns out that there is exactly the same statement about  $\tau_u \Delta_v \tau_u^{-1}$  as we had about  $\Delta_v^{-1} \tau_u \Delta_v$ . To proceed we need a (partially defined) operator  $S^{-1}$  which acts on operators. Set  $\tau = \tau_1$ ,  $M = (1 - q)(1 - t)$ ,

$$\tau F = F[X + 1], \quad \tau^* F = \text{Exp} \left[ -\frac{X}{M} \right],$$

where  $\text{Exp}$  is the plethystic exponential,

$$\text{Exp}[X] = \sum_{n=0}^{\infty} h_n[X] = \exp \left( \sum_{n=1}^{\infty} \frac{p_n[X]}{n} \right).$$

Strictly speaking,  $\tau^*$  sends a symmetric function to an infinite series so that the degrees of the terms tend to infinity. Such infinite series form a complete topological

algebra  $\text{Sym}[[X]]$  in a standard way, with the algebra of symmetric functions being a dense subalgebra  $\text{Sym}[X]$ . Thus it makes sense to speak of continuous operators acting on symmetric functions.

**Proposition 2.4.** *Let  $L$  be a continuous linear operator acting on the space of symmetric functions. There exists at most one continuous linear operator  $S^{-1}(L)$ <sup>2</sup> such that for all  $F \in \text{Sym}[X]$*

$$\tau^* \tau L F = S^{-1}(L) \tau^* \tau F.$$

*The set of all continuous operators such that  $S^{-1}(L)$  exists form an algebra, and  $S^{-1}$  is an algebra homomorphism.*

*Proof.* The statement follows from the fact that the image of the operator  $\tau^* \tau : \text{Sym}[X] \rightarrow \text{Sym}[[X]]$  is dense.  $\square$

Now we can formulate our statement

**Proposition 2.5.** *For a symmetric function  $F$  of degree  $d$  the operator*

$$\tau_u \Delta_F \tau_u^{-1}$$

*is a polynomial in  $u$  of degree  $\leq d$ . The coefficient of  $u^d$  is the operator  $S^{-1}(\Delta'_F)$ .*

*Proof.* For the first part of the statement we can replace  $\Delta_F$  by  $\Delta'_F$ . It is well known (see, for instance, [BGLX14]) that the operator  $\Delta'_F$  can be written as a linear combination of the operators  $D_{i_1} D_{i_2} \cdots D_{i_d}$ , where  $D_n$  is defined for any  $n \in \mathbb{Z}$  as

$$D_n = F[X + Mz^{-1}] \text{Exp}[-Xz] \Big|_{z^n}.$$

Thus, it is enough to prove the statement for the operators  $D_n$ . We consider  $\tau_u D_n \tau_u^{-1}$ :

$$F[X] \rightarrow F[X-u] \rightarrow F[X+Mz^{-1}-u] \text{Exp}[-Xz] \Big|_{z^n} \rightarrow F[X+Mz^{-1}] \text{Exp}[-Xz-uz] \Big|_{z^n}.$$

Because  $\text{Exp}[-uz] = 1 - uz$ , we obtain

$$\tau_u D_n \tau_u^{-1} = D_n - u D_{n-1}.$$

So we see that we obtained a polynomial in  $u$  of degree  $\leq 1$ , and the top coefficient is  $-D_{n-1}$ . It remains to show that  $S^{-1} D_n = -D_{n-1}$ . Note that  $D_n$  is homogeneous of degree  $n$ , therefore continuous for all  $n$ . Comparing

$$\tau^* \tau D_n F = F[X + Mz^{-1} + 1] \text{Exp} \left[ -Xz - z - \frac{X}{M} \right] \Big|_{z^n}$$

<sup>2</sup>The operation  $S^{-1}$ , after a sign change turns out to be inverse to  $S$  from [BGLX14], [BGLX15]

$$= F[X + Mz^{-1} + 1] \text{Exp} \left[ -Xz - \frac{X}{M} \right] \Big|_{z^n} - F[X + Mz^{-1} + 1] \text{Exp} \left[ -Xz - \frac{X}{M} \right] \Big|_{z^{n-1}}$$

and

$$\begin{aligned} -D_{n-1}\tau^*\tau F &= -F[X + Mz^{-1} + 1] \text{Exp} \left[ -Xz - z^{-1} - \frac{X}{M} \right] \Big|_{z^{n-1}} \\ &= -F[X + Mz^{-1} + 1] \text{Exp} \left[ -Xz - \frac{X}{M} \right] \Big|_{z^{n-1}} + F[X + Mz^{-1} + 1] \text{Exp} \left[ -Xz - \frac{X}{M} \right] \Big|_{z^n} \end{aligned}$$

we see that  $S^{-1}D_n$  exists and equals  $-D_{n-1}$   $\square$

Proposition implies that in each term of  $\tau_u \Delta_v \tau_u^{-1}$  the exponent of  $u$  is less or equal to the exponent of  $v$ . Multiplication by  $\Delta_v^{-1}$  on the right only increases the exponent of  $v$ . This proves the following

**Proposition 2.6.** *We have  $W_{i,j} = 0$  for  $i < j$  and  $W_{i,i} = S^{-1}(\Delta'_{e_i})$ .*

Putting Propositions 2.3 and 2.6 together we obtain

**Theorem 2.7.** *The commutator of  $\Delta_v^{-1}$  and  $\tau_u$  has the following two expressions:*

$$\Delta_v^{-1} \tau_u \Delta_v \tau_u^{-1} = \nabla^{-1} \tau_{uv} \nabla = S^{-1}(\Delta'_{uv}).$$

This establishes Conjecture 6 of [BH13]. To complete our proof of Theorem 1.1 denote for any operator  $L$

$$N(L) = \nabla L \nabla^{-1}, \quad N^{-1}(L) = \nabla^{-1} L \nabla.$$

Let also  $N^{-1}$  send  $u, v$  to  $uv, v$  respectively, and let  $S^{-1}$  send  $(u, v)$  to  $u, uv$  respectively. We define  $R_{k,0}$ ,  $R_{0,k}$  and then  $T_{0,1}$ ,  $T_{1,0}$  as in (3), so that

$$T_{1,0} = \tau_u, \quad T_{0,1} = \Delta'_v,$$

$$(7) \quad T_{0,1}^{-1} T_{1,0} T_{0,1} T_{1,0}^{-1} = N^{-1}(T_{1,0}) = S^{-1}(T_{0,1}).$$

Hence we must necessarily have

$$T_{1,1} = N^{-1}(T_{1,0}) = S^{-1}(T_{0,1}).$$

Moreover, we have

$$(8) \quad N^{-1}(T_{0,1}) = T_{0,1}, \quad S^{-1}(T_{1,0}) = \text{Exp} \left[ \frac{u}{M} \right] T_{1,0}.$$

From this data there is a unique way to construct operators  $T_{m,n}$  for all relatively prime pairs  $m, n$  such that

$$N^{-1}(T_{m,n}) = T_{m,m+n}, \quad S^{-1}(T_{m,n}) = T_{m+n,n} \quad (n > 0).$$

Moreover, applying the operators  $N^{-1}$ ,  $S^{-1}$  to (7) we obtain the statements (1) for all  $m, n, m', n'$  with  $mn' - m'n = 1$ . The statement for  $R_{k,0}$ ,  $R_{0,k}$  as in (4) is obtained from the statement we just established by applying the conjugation with respect to the modified Hall inner product on  $\text{Sym}[X]$  defined as

$$(F[X], G[X])_* := (F[-MX], G[X]),$$

where in the right hand side we have the standard Hall scalar product.

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